Conformal field theory correlations in the Abelian sandpile model

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We calculate all multipoint correlation functions of all local bond modifications in the two-dimensional Abelian sandpile model, both at the critical point, and in the model with dissipation. The set of local bond modifications includes, as the most physically interesting case, all weakly allowed cluster variables. The correlation functions show that all local bond modifications have scaling dimension 2, and can be written as linear combinations of operators in the central charge −2 logarithmic conformal field theory, in agreement with a form conjectured earlier by Mahieu and Ruelle in Phys. Rev. E **64**, 066130 (2001). We find closed form expressions for the coefficients of the operators, and describe methods that allow their rapid calculation. We determine the fields associated with adding or removing bonds, both in the bulk, and along open and closed boundaries; some bond defects have scaling dimension 2, while others have scaling dimension 4. We also determine the corrections to bulk probabilities for local bond modifications near open and closed boundaries.

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I. INTRODUCTION

Self-organized criticality may be the underlying cause of power laws in a wide range of natural and man-made phenomena [1,2]. Systems exhibiting self-organized criticality naturally approach a critical state, without any intrinsic time or distance scales. The critical point is reached without any fine-tuning of parameters. This is unlike most critical points seen in physics—for example, the Ising model is only critical at a single, very specific, temperature. This lack of fine-tuning is essential if we are to understand power laws in nature, where no fine-tuning is possible.

Since the concept of self-organized criticality was introduced by Bak, Tang, and Wiesenfeld in 1987, a number of models have been developed to investigate this phenomenon [3]. However, the original model, the two-dimensional, isotropic, Abelian sandpile model (ASM), is still one of the simplest and most interesting of the models. The ASM is simple and robust, which are necessary features for any model of self-organized criticality. While natural phenomena are quite complex, any model that seeks to explain the ubiquity of power laws in nature must, paradoxically, be very simple; if we are to have a robust model for the generation of power laws, we must neither have finely tuned parameters, nor finely tuned rules.

The ASM is defined on a lattice of sites, and is described by a toppling matrix $\mathbf{\Delta}$, whose dimension is equal to the number of sites in the sandpile. The sandpile evolves stochastically. In each time step, a grain of sand is added to a random site. Then, sites are checked for stability. If the number of grains at a site $\vec{i}$ is greater than $\Delta_{\vec{i},\vec{i}}>0$, then the site $\vec{i}$ is unstable, and topples, losing $\Delta_{\vec{i},\vec{i}}$ grains, while every other site $\vec{j}$ gains $-\Delta_{\vec{i},\vec{j}}\geq 0$ grains. (Generally, $\Delta_{\vec{i},\vec{j}}$ is zero except when $\vec{j}$ neighbors $\vec{i}$.) Typically, models are conservative, which means that each toppling in the bulk conserves the total number of grains ($\Sigma_{\vec{j}}\Delta_{\vec{i},\vec{j}}=0$). Only for topplings along the boundary, where grains can fall off the edge, can the total number of grains change. We continue toppling unstable sites until no sites are unstable. Then, we begin a new time step, and again add a grain to a random site.

The ASM is surprisingly tractable [4–6]. We only briefly cover some of the essential points here—for comprehensive reviews, see Refs. [7,8].

After a large number of time steps, the ASM reaches a well-defined distribution of states. Of the stable height configurations, some are transient, and occur with probability zero after a long amount of time. All other states are recurrent, and occur with equal probability. Dhar showed that the total number of recurrent states is just $\det(\mathbf{\Delta})$ [5]. This is also equal to the number of spanning trees that can be drawn on the lattice, showing a connection between the sandpile and spanning tree problems [6].

These statements hold for all ASMs, which define a large class of models. Now, we specialize to the two-dimensional, conservative, isotropic ASM, which is defined on a two-dimensional square lattice, where each site has a maximum height of 4, and where upon toppling at any site, one grain is sent to each of the site's four neighbors. Furthermore, we work in the limit where the lattice is infinite. The two-dimensional, isotropic, spanning tree problem is equivalent to the central charge −2 logarithmic conformal field theory ($c=-2$ LCFT) [6], which has the simple Gaussian action $S=(1/\pi)\int \partial\theta\bar{\partial}\bar{\theta}$, where θ and $\bar{\theta}$ are complex Grassman variables. The $c=-2$ LCFT is described in Refs. [9–11]. While the two-dimensional, conservative, isotropic ASM is just one of many possible ASMs, it is the original, standard, model [3], and it is reasonable to simply refer to it as "the ASM," which we do for the remainder of this paper.

Calculations of correlation functions, using methods to be described in the next section, have confirmed that there is a relationship between the ASM and $c=-2$ LCFT. Two-point correlation functions of unit height variables decay as $1/r^4$ in the bulk [4], as do all two-point height correlations along open and closed boundaries [12]; these correlations can be

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understood as equivalent to correlations of LCFT operators. Furthermore, calculations of certain three-point correlation functions of heights along closed boundaries, and all multipoint correlations of heights along open boundaries, have allowed LCFT field identifications for heights along boundaries [13,14].

The ASM is not robust to all perturbations. If we relax the constraint that the model be conservative, and instead allow grains to be lost in any bulk toppling (i.e., allow dissipation), correlations decay exponentially, and we are taken off the critical point [15–17]. The condition of conservation can be considered a "natural" one, rather than one requiring "fine-tuning." Deeper probes of the the conformal structure can be obtained by looking at correlations both on and off the critical point.

Mahieu and Ruelle calculated a number of off-critical correlation functions of certain height configurations, known as weakly allowed clusters (WACs), and used their correlation functions to propose field identifications for the 14 simplest WACs [18]. They found that their correlation functions could be explained by assuming that all 14 WACs took the form

$$\phi(z) = -\left\{ A{:}\partial\theta\bar{\partial}\bar{\theta} + \bar{\partial}\theta\partial\bar{\theta}{:} + B_1{:}\partial\theta\partial\theta + \bar{\partial}\bar{\theta}\bar{\partial}\bar{\theta}{:} + iB_2{:}\partial\theta\bar{\partial}\theta - \bar{\partial}\theta\bar{\partial}\bar{\theta}{:} + \frac{CP(S)M^2}{2\pi}{:}\theta\bar{\theta}{:} \right\}. \quad (1)$$

The coefficients A, B_1, B_2, C, and $P(S)$ vary from WAC to WAC. $P(S)$ is the probability for the cluster at the critical point, and M is the mass, a measure of how far the model is from the critical point. The correlation functions that they used were mostly two-point functions along horizontal or diagonal axes, as well as some three-point and four-point functions for the two simplest WACs.

While these calculations provide strong evidence for the identification of the ASM with the $c=-2$ LCFT, and the field identification in Eq. (1), the fact that only specific correlations were considered limits the range of the identification. It would be surprising if new orientations of correlation functions, or new WACs, were found to be inconsistent with Eq. (1); but the calculations in Ref. [18] do not rule this possibility out. More importantly, since each correlation function in Ref. [18] required a new and separate calculation, it is hard to understand, mathematically, why these results occurred. While their end results showed that certain correlations of WACs in the ASM are equal to correlations of Eq. (1) in the LCFT, it was not mathematically transparent as to why this should be. Nor was it clear why, or if, the same coefficients would appear in other properties, such as off-boundary correlations, or correlations with defects.

Here, we calculate all correlation functions of all local bond modifications (LBMs), for arbitrary numbers and types of LBMs at arbitrary positions (far from one another); our calculations confirm that all LBMs should receive the field identification in Eq. (1). By LBMs, we mean any set of local changes in the sandpile toppling rules. For the ASM at the critical point, we will assume that the LBMs are conservative (do not create or destroy grains). Since all WACs can be calculated by LBMs, our results automatically include all correlations of WACs. While the WACs are the most important types of LBMs, and the easiest to find probabilities of, in numerical simulations, we generally discuss our results in terms of LBMs, to emphasize the generality of our results. We give closed form expressions for A, B_1, B_2, and C, and describe methods that allow rapid calculation of these coefficients. While a computer is needed for the calculation of specific A, B_1, and B_2 coefficients, the general calculations can be done by hand.

By showing how calculation for all LBMs can be done at once, we make the mathematical structure clearer. For example, we can quickly see why the coefficients A, B_1, and B_2 appear in other properties. We illustrate this by looking at off-boundary LBM probabilities, and correlations with bond defects (either in the bulk, or along a boundary). Interestingly, we find that some bond defects are represented by a LCFT operator with a scaling dimension of 4. Until now, all (nonderivative) fields in the ASM have been found to be dimension 0 or 2.

While our calculations have been done in both the normal ASM and the ASM with dissipation, we focus our discussion on the simpler analysis at the critical point, and only discuss the more complicated massive correlations in the last section, and in the appendixes.

II. WEAKLY ALLOWED CLUSTER VARIABLES

The methods used in this paper are not powerful enough to calculate probabilities and correlations for any height configurations. Even the calculation of the probability for a site to have height 2 requires much more complicated methods [19], and the correlation function of two height 2 variables remains unknown. This is because the condition for a site to have height 2 involves a nonlocal condition.

As already stated, the most important LBMs are those used to calculate properties of WACs [20]. WACs are related to forbidden subconfigurations (FSCs). An FSC is a height configuration over a subset of sites F, such that for every site $\vec{i} \in F$, the number of neighbors of $\vec{i}$ in F is greater than or equal to the height at $\vec{i}$. FSCs are important because ASM height configurations are recurrent if and only if they have no FSCs [5]. A WAC is a height configuration that contains no FSCs, but becomes a FSC if any height in the WAC is decreased by 1. Three WACs are shown on the left side of Fig. 1.

WACs are analytically tractable because it turns out that the number of sandpiles with a particular WAC is equal to the number of recurrent states in a sandpile with modified toppling rules [4]. There are actually several different ways to modify the toppling rules to obtain the WAC probability. The simplest is, for each connected piece of the WAC, to remove all but one of the bonds connecting it to the rest of the lattice—the modified lattices corresponding to the WACs are shown on the right side of Fig. 1. (See Ref. [18] for a discussion of other ways in which the sandpile can be modified to obtain the WAC probabilities.) In these modified sandpiles, grains of sand cannot flow along the removed bonds; to continue to conserve the number of grains during each toppling, the condition for instability must be decreased

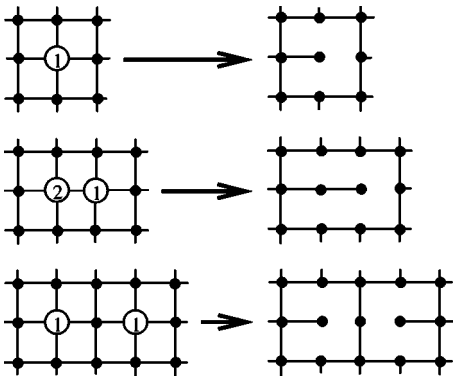

FIG. 1. Some WACs and their corresponding modified sandpiles.

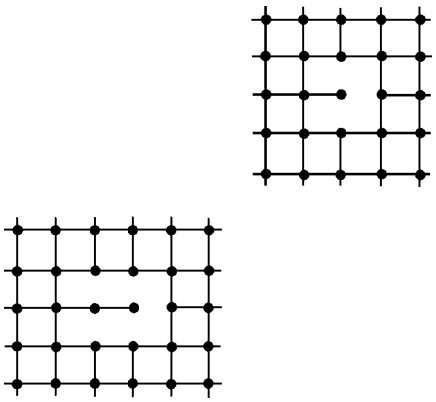

FIG. 2. Modified sandpile for a WAC two-point correlation.

at the sites at the end of the removed bond. These changes result in a new toppling matrix $\mathbf{\Delta}'$.

As already stated, the number of recurrent states in the ASM is $\det(\mathbf{\Delta})$. The number of recurrent states that have the WAC is given by $\det(\mathbf{\Delta}')$. (We discuss this equivalence further in Sec. V.) So the bulk probability for the WAC is given by

$$p = \frac{\det(\mathbf{\Delta}')}{\det(\mathbf{\Delta})} = \det(\mathbb{I} + \mathbf{BG}), \tag{2}$$

where we have defined $\mathbf{B} = \mathbf{\Delta}' - \mathbf{\Delta}$ and $\mathbf{G} = \mathbf{\Delta}^{-1}$. $\mathbf{G}$ is the well-studied lattice Green function (at the critical point); exact expressions are known for the Green function between nearby sites, and asymptotic expressions for the Green function between distant sites [21]. While $\mathbf{\Delta}, \mathbf{\Delta}'$, and $\mathbf{G}$ all have large dimensions (equal to the number of sites), $\mathbf{B}$ is zero outside of a finite collection of sites. When the bond between $\vec{i}$ and $\vec{j}$ is removed, $B_{\vec{i},\vec{j}}$ and $B_{\vec{j},\vec{i}}$ are both increased by 1, while $B_{\vec{i},\vec{i}}$ and $B_{\vec{j},\vec{j}}$ are both decreased by 1. For example, for the unit height probability, we have

$$\mathbf{B} = \begin{array}{c} \begin{matrix} \vec{i} & \vec{j}_1 & \vec{j}_2 & \vec{j}_3 \end{matrix} \\ \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{matrix} \vec{i} \\ \vec{j}_1 \\ \vec{j}_2 \\ \vec{j}_3 \end{matrix} \tag{3}$$

Here $\vec{i}$ is the site fixed at height 1, while $\vec{j}_1, \vec{j}_2$, and $\vec{j}_3$ are the three sites that $\vec{i}$ has been disconnected from.

For any WAC, the fact that $\mathbf{B}$ is finite-dimensional means that the height probability can be found by calculating a simple, finite-dimensional, matrix determinant. All WACs thus correspond to LBMs. However, many LBMs do not correspond to WACs. LBMs are simply any sandpile modifications that can be modeled with a $\mathbf{B}$ matrix that is conservative (every row and column sums to zero) and symmetric. Our analysis gives all correlations of LBMs, which thus automatically gives all correlations of WACs.

III. CORRELATIONS OF LOCAL BOND MODIFICATIONS

For an n-point correlation function of LBMs, we can still use this method. The only difference is that the removed bonds are located in n distant clusters; this is illustrated in Fig. 2. Removal of bonds in this fashion will give $\mathbf{B}$ and $\mathbf{G}$ block matrix structures. For example, for a three-point function, we will have

$$\mathbf{B} = \begin{pmatrix} \mathbf{B_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B_3} \end{pmatrix}, \tag{4}$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{G_{11}} & \mathbf{G_{12}} & \mathbf{G_{13}} \\ \mathbf{G_{21}} & \mathbf{G_{22}} & \mathbf{G_{23}} \\ \mathbf{G_{31}} & \mathbf{G_{32}} & \mathbf{G_{33}} \end{pmatrix}. \tag{5}$$

$\mathbf{B}_u$ is the modification to the toppling matrix for the set of bonds removed about the uth LBM. $\mathbf{G}_{uu}$ is the Green function matrix between sites of the uth LBM, and its elements are $O(1)$. $\mathbf{G}_{uv}, u \neq v$ is the Green function matrix between the sites of the uth and vth LBMs, and its elements are given by the bulk Green function,

$$G_0(x,y) = -\frac{1}{4\pi}\ln(x^2+y^2) - \frac{\gamma}{2\pi} - \frac{\ln 8}{4\pi} + \cdots, \tag{6}$$

where $\gamma = 0.577\,21\cdots$ is the Euler-Mascheroni constant [21]. We work in the limit where the LBMs are all very far from each other—we assume that any two of the n LBMs are the same order of magnitude, $O(r)$, apart. Since the Green function diverges as $\ln(r)$ with increasing r, calculation of $\det(\mathbb{I}+\mathbf{BG})$ initially looks very difficult. However, every row of every $\mathbf{B}_u$ sums to zero—this follows from the manner in which we constructed $\mathbf{B}_u$, and reflects the fact that grains of sand are still conserved in each toppling in the bulk of the modified sandpile. This implies that parts of $\mathbf{G}_{uv}$ that depend only on the column index make no contribution to $\mathbf{BG}$, and thus no contribution to the correlation function, $\det(\mathbb{I}+\mathbf{BG})$. So we only care about differences (discrete derivatives) of Green functions between columns of $\mathbf{G}_{uv}$, and the elements of $\mathbf{G}_{uv}$ are effectively $O(1/r)$, rather than $O(\ln r)$.

For LBMs, every $\mathbf{B}_u$ is symmetric, so every column of every $\mathbf{B}_u$ sums to zero. Using the matrix identity

$\det(\mathbb{I}+\mathbf{BG})=\det(\mathbb{I}+\mathbf{GB})$, this in turn means that the parts of $\mathbf{G}_{uv}$ that depend only on the row index make no contribution to the probability. This is, in effect, like taking another discrete derivative of the Green function, so that the elements of $\mathbf{G}_{uv}$ are effectively $O(1/r^2)$.

To make this concrete, suppose that the local origin of the uth LBM is located at (0, 0), and the local origin of the vth LBM is located at $(x_{uv},y_{uv})=(r_{uv}\cos\phi_{uv}, r_{uv}\sin\phi_{uv})$. The uth LBM covers a set of sites at locations (k_1,l_1), relative to (0, 0), and the vth LBM consists of a series of sites at locations (k_2,l_2), relative to (x_{uv},y_{uv}). [k_1,k_2,l_1, and l_2 are all $O(1)$.] Then, the elements of $\mathbf{G}_{uv}$ all have the form $G_0(x_{uv}+k_2-k_1, y_{uv}+l_2-l_1)$. The last two paragraphs show that we only need the parts of $\mathbf{G}_{uv}$ that depend on *both* the row *and* column indices. That is, we only need the parts of the Green function that depend on *both* (k_1,l_1) *and* (k_2,l_2), and can drop all other terms. Expanding Eq. (6) in powers of $1/r_{uv}$, we find that the lowest-order term not dropped is

$$G_0(x_{uv}+k_2-k_1,y_{uv}+l_2-l_1) \to -\frac{1}{2\pi r_{uv}^2}[(k_1k_2-l_1l_2)\cos(2\phi_{uv}) + (k_1l_2+k_2l_1)\sin(2\phi_{uv})]. \tag{7}$$

The Green function can thus be treated as $O(1/r^2)$ for correlations of LBMs. For more general local arrow diagrams, such as those that appear in the calculations involving the height two variable [12,19], the **B** matrices are not symmetric, and we can no longer drop the parts of $\mathbf{G}_{uv}$ that depend only on the row index.

To get the connected n-point function from $\det(\mathbb{I}+\mathbf{BG})$, we need to pick at least one element off the block diagonal in every block row of **G**, and in every block column of **G**. This means, at the minimum, picking n elements off the block diagonal of **G**, resulting in a leading-order contribution to the correlation function of $O(1/r^{2n})$—this is the universal part of the correlation function.

Mahieu and Ruelle showed that for two-point functions, the constraint of picking only two elements off the block diagonal allows the correlation function to be written as [18]

$$\det(\mathbb{I}+\mathbf{BG}) = -p_{u_1}p_{u_2}\,\mathrm{Tr}\left\{\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_1}\mathbf{G}_{u_1u_1}}\times\mathbf{B}_{u_1}\mathbf{G}_{u_1u_2}\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_2}\mathbf{G}_{u_2u_2}}\mathbf{B}_{u_2}\mathbf{G}_{u_2u_1}\right\}. \tag{8}$$

Similarly, they found that the leading-order contribution to the three-point probability is

$$\det(\mathbb{I}+\mathbf{BG}) = p_{u_1}p_{u_2}p_{u_3}\,\mathrm{Tr}\left\{\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_1}\mathbf{G}_{u_1u_1}}\mathbf{B}_{u_1}\mathbf{G}_{u_1u_2}\times\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_2}\mathbf{G}_{u_2u_2}}\mathbf{B}_{u_2}\mathbf{G}_{u_2u_3}\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_3}\mathbf{G}_{u_3u_3}}\mathbf{B}_{u_3}\mathbf{G}_{u_3u_1}\right\} + p_{u_1}p_{u_2}p_{u_3}\,\mathrm{Tr}\left\{\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_1}\mathbf{G}_{u_1u_1}}\mathbf{B}_{u_1}\mathbf{G}_{u_1u_3}\times\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_3}\mathbf{G}_{u_3u_2}}\mathbf{B}_{u_3}\mathbf{G}_{u_3u_2}\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_2}\mathbf{G}_{u_2u_2}}\mathbf{B}_{u_2}\mathbf{G}_{u_2u_1}\right\}. \tag{9}$$

More generally, for an n-point correlation, if only n terms are picked off the block diagonal, then the connected correlation function is given by

$$\det(\mathbb{I}+\mathbf{BG}) = \frac{(-1)^{n+1}}{n}\left[\prod_{x=1}^{n}p_{u_x}\right]\times\sum_s\,\mathrm{Tr}\left\{\prod_{x=1}^{n}\left[\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_{u_{s(x)}}\mathbf{G}_{u_{s(x)}u_{s(x)}}}\times\mathbf{B}_{u_{s(x)}}\mathbf{G}_{u_{s(x)}u_{s(x+1)}}\right]\right\}, \tag{10}$$

where s is summed over all one-to-one mappings from $\{1,2,\ldots n\}$ to $\{1,2,\ldots n\}$, and we identify $s(n+1)$ with $s(1)$. Mahieu and Ruelle wrote Eqs. (8) and (9) in different, but equivalent, forms.

$$p_u = \det(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}) \tag{11}$$

is the bulk probability of the uth LBM. [Note that the two trace terms of Eq. (9) are actually equal. We have written the three-point function in this form to make clear how the form generalizes for n-point functions.]

We can rewrite Eq. (7) as

$$\mathbf{G}_{uv} = -\frac{1}{2\pi r_{uv}^2}[(\vec{k}_u\vec{k}_v^T-\vec{l}_u\vec{l}_v^T)\cos(2\phi_{uv}) + (\vec{k}_u\vec{l}_v^T+\vec{l}_u\vec{k}_v^T)\sin(2\phi_{uv})]. \tag{12}$$

$\vec{k}_u$ is the column vector of the horizontal positions of the sites of the uth LBM, relative to the uth local origin [i.e., the elements of $\vec{k}_u$ are $O(1)$]. $\vec{l}_u$ is the corresponding vector of vertical positions. $\vec{k}_u$ and $\vec{l}_u$ are both length N_u, where N_u is the number of sites needed to represent the uth LBM with the methods of the previous section (e.g., $N_u=4$ for the unit height variable).

We insert Eq. (12) into Eq. (10). For each of the n $\mathbf{G}_{uv}$'s, we can pick any of the four matrices of Eq. (12), resulting in 4^n terms. In each of these 4^n terms, each $\mathbf{G}_{uv}$ has been replaced with the product of a column vector and a row vector. Using the cyclicity of the trace to move one row vector at the end of the trace to the start of the trace, we see that each matrix $(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu})^{-1}\mathbf{B}_u$ is bracketed by a row vector to its left, and a column vector to its right, producing a 1×1 matrix. So each of the 4^n terms is the product of n numbers. We can represent the decisions as to which terms of Eq. (12) to pick by representing $\mathbf{G}_{uv}$ with a 2×2 matrix, $\mathbf{N}_{uv}$. The possible ways to bracket $(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu})^{-1}\mathbf{B}_u$ can be represented with a 2×2 matrix, $\mathbf{M}_u$. We have

$$\mathbf{M}_u \equiv \begin{array}{c} \begin{matrix} \vec{k}_u^{\,T} & \vec{l}_u^{\,T} \end{matrix} \\ \begin{pmatrix} c_{u,kk} & c_{u,kl} \\ c_{u,kl} & c_{u,ll} \end{pmatrix} \begin{matrix} \vec{k}_u \\ \vec{l}_u \end{matrix} \end{array}, \tag{13}$$

$$\mathbf{N}_{uv} \equiv -\frac{1}{2\pi r_{uv}^2} \begin{array}{c} \begin{matrix} \vec{k}_v^{\,T} & \vec{l}_v^{\,T} \end{matrix} \\ \begin{pmatrix} \cos(2\phi_{uv}) & \sin(2\phi_{uv}) \\ \sin(2\phi_{uv}) & -\cos(2\phi_{uv}) \end{pmatrix} \begin{matrix} \vec{k}_u \\ \vec{l}_u \end{matrix} \end{array}. \tag{14}$$

We have defined

$$c_{u,kk} \equiv -p_u \vec{k}_u^T \frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}} \mathbf{B}_u \vec{k}_u, \tag{15}$$

$$c_{u,kl} \equiv -p_u \vec{k}_u^T \frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}} \mathbf{B}_u \vec{l}_u = -p_u \vec{l}_u^T \frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}} \mathbf{B}_u \vec{k}_u, \tag{16}$$

$$c_{u,ll} \equiv -p_u \vec{l}_u^T \frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}} \mathbf{B}_u \vec{l}_u. \tag{17}$$

Then, the correlation function of n LBMs is given by

$$-\mathrm{Tr}(\mathbf{M}_{u_1}\mathbf{N}_{u_1u_2}\mathbf{M}_{u_2}\mathbf{N}_{u_2u_3}\cdots\mathbf{M}_{u_n}\mathbf{N}_{u_nu_1}) - \{[(n-1)!-1]\ (\text{other trace terms})\}, \tag{18}$$

where the other trace terms are derived by permutations of $\{u_1, u_2, \ldots, u_n\}$, as in Eq. (10).

We can compare this to correlation functions of fields in the $c=-2$ LCFT. Mahieu and Ruelle proposed that the WACs are represented, at the critical point, by

$$\phi_u(z_u) = -\{A_u{:}\partial\theta\bar{\partial}\bar{\theta}{+}\bar{\partial}\theta\,\partial\bar{\theta}{:}+B_{1u}{:}\partial\theta\partial\bar{\theta}{+}\bar{\partial}\theta\bar{\partial}\bar{\theta}{:} + iB_{2u}{:}\partial\theta\partial\bar{\theta}{-}\bar{\partial}\theta\bar{\partial}\bar{\theta}{:}\}. \tag{19}$$

[The "C" term in Eq. (1) only appears off the critical point.]

We can compute connected n-point correlations of these fields in the $c=-2$ LCFT. We use the formulation of the $c=-2$ LCFT where the action is

$$S = \frac{1}{\pi}\int d^2x\,{:}\partial\theta\bar{\partial}\bar{\theta}{:}, \tag{20}$$

where we do not integrate over zero modes in expectation values. Since the theory is Gaussian, to calculate correlation functions we simply need to take Wick contractions. The relevant nonzero ones are

$$\langle\partial\theta(z_u)\bar{\partial}\bar{\theta}(z_v)\rangle = -\frac{1}{2(z_u-z_v)^2} = -\frac{e^{-2i\phi_{uv}}}{2r_{uv}^2}, \tag{21}$$

$$\langle\bar{\partial}\theta(z_u)\bar{\partial}\bar{\theta}(z_v)\rangle = -\frac{1}{2(\bar{z}_u-\bar{z}_v)^2} = -\frac{e^{+2i\phi_{uv}}}{2r_{uv}^2}. \tag{22}$$

Each term of Eq. (19) has one θ, and one $\bar{\theta}$. The only difference between terms is whether the derivative on the θ is holomorphic or antiholomorhic, and whether the derivative on the $\bar{\theta}$ is holomorphic or antiholomorphic. We can use a 2×2 matrix to represent the choice of which terms of $\phi_u(z_u)$ are picked:

$$\mathbf{F}_u = \frac{\partial\theta}{\bar{\partial}\bar{\theta}} \begin{array}{c} \begin{matrix} \partial\bar{\theta} & \bar{\partial}\bar{\theta} \end{matrix} \\ \begin{pmatrix} B_{1u}+iB_{2u} & A_u \\ A_u & B_{1u}-iB_{2u} \end{pmatrix} \end{array}. \tag{23}$$

The contractions of Eqs. (21) and (22) can then be represented with the matrix

$$\mathbf{H}_{uv} = \frac{\partial\bar{\theta}}{\bar{\partial}\bar{\theta}} \begin{array}{c} \begin{matrix} \partial\theta & \bar{\partial}\theta \end{matrix} \\ \begin{pmatrix} -e^{-2i\theta_{uv}}/(2r_{uv}^2) & 0 \\ 0 & -e^{+2i\theta_{uv}}/(2r_{uv}^2) \end{pmatrix} \end{array}. \tag{24}$$

The contribution to the correlation function where the $\bar{\theta}$ of the first LBM contracts with the θ of the second LBM, the $\bar{\theta}$ from the second LBM contracts with the θ of the third LBM, and so on, is

$$-\mathrm{Tr}(\mathbf{F}_{u_1}\mathbf{H}_{u_1u_2}\mathbf{F}_{u_2}\mathbf{H}_{u_2u_3}\cdots\mathbf{F}_{u_n}\mathbf{H}_{u_nu_1}). \tag{25}$$

Other contractions give other permutations, just as in Eq. (18). Finally, $\mathbf{M}_u\mathbf{N}_{uv}$ differs from $\mathbf{F}_u\mathbf{G}_{uv}$ only by a matrix rotation, which will not affect the trace, if we make the following identifications:

$$A_u = \frac{1}{2\pi}(c_{u,kk}+c_{u,ll}), \tag{26}$$

$$B_{1u} = \frac{1}{2\pi}(c_{u,kk}-c_{u,ll}), \tag{27}$$

$$B_{2u} = \frac{1}{\pi}c_{u,kl}. \tag{28}$$

So the traces in Eqs. (18) and (25) are equal, and all LBMs are indeed represented by the field in Eq. (19). These formulas for the coefficients have the appropriate transformation properties under 90° rotations, and x and y reflections. [Technically, the overall sign of Eq. (26) is still undetermined at this point, since all correlation functions have even numbers of A's. To determine the signs of the A's we need to look at at least one massive correlation function. We can do this by consulting the massive three-point function of the unit height variable in Ref. [18], or more broadly, by looking at the general massive correlations in Sec. IX.]

IV. COMPUTATION OF A, B_1, AND B_2 TERMS

A, B_1, and B_2, can be calculated on a computer with Eqs. (26)–(28) and Eqs. (15)–(17). Evaluating Eqs. (15)–(17), as written, requires taking a matrix inverse, which can be computationally time consuming for larger LBMs. The calculation can be made substantially faster with the following matrix identity, which we state without proof:

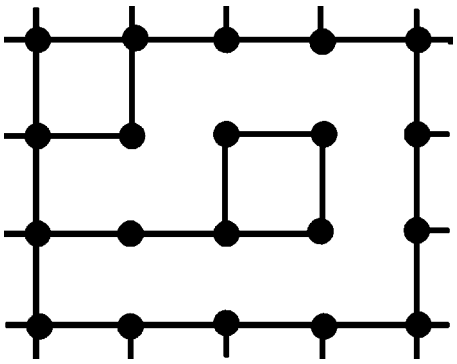

FIG. 3. Modified sandpile for the height configurations in Figs. 4 and 5.

$$\det[\mathbb{I} + \mathbf{B}(\mathbf{G} + K\vec{f}\vec{g}^T)] = \det(\mathbb{I} + \mathbf{BG}) + K \det(\mathbb{I} + \mathbf{BG}) \times \left(\vec{g}^T \frac{\mathbb{I}}{\mathbb{I} + \mathbf{BG}} \mathbf{B}\vec{f} \right). \quad (29)$$

This identity holds for any vectors $\vec{f}$ and $\vec{g}$, and c -number K. It allows us to compute the c's in Eqs. (15)–(17) as matrix determinants, which is faster than computing matrix inverses. Furthermore, we note that, in general, certain combinations of rows (and columns) of $\mathbf{B}_u$ will sum to zero, which means that we can perform a matrix rotation to reduce the size of the matrix determinant. With these methods, computation of A, B_1, and B_2 for the ten simplest WACs takes roughly one hour, using MATHEMATICA on a computer with a 1.2-GHz processor. The results agree with those found in Ref. [18]. Comparison with two of the larger WACs, which they label S_{10} and S_{11}, requires a more detailed discussion of the mapping between WACs and LBMs, which is done in the next section.

V. MAPPING BETWEEN WEAKLY ALLOWED CLUSTER AND LOCAL BOND MODIFICATIONS

We illustrate the mapping between larger weakly allowed clusters and local bond modifications with the sandpile modification shown in Fig. 3. In this modified sandpile, a five-site cluster is separated from the rest of the sandpile, except by a single bond. The number of states in the modified sandpile of Fig. 3 is equal to the number of states of the unmodified ASM where decreasing the five-site cluster's left-most site (which we call $\vec{i}$) from 2 to 1 makes the five-site cluster a FSC, *and* does not produce any larger FSCs [19]. The condition that the FSC produced be maximal is necessary for this equivalence, although this condition was not explicitly stated in Ref. [19]. If changing the height of $\vec{i}$ from 2 to 1 makes the five-site cluster a FSC, the original height configuration (before this change) must have been one of the four configurations shown in Figs. 4 and 5. Of these four configurations, the one in Fig. 4 is not a WAC, while the three in Fig. 5 are.

Absent other conditions, the configuration in Fig. 4 does not have the same probability as the configurations in Fig. 5. The configuration in Fig. 4 is more likely, as an allowed

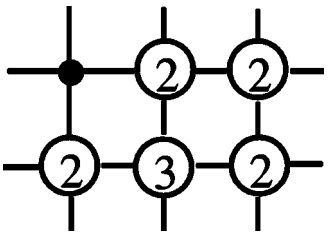

FIG. 4. A non-WAC with the same shape as the configuration in Fig. 3.

configuration always stays allowed when a site height is increased. However, for some configurations of heights outside the five-site cluster, decreasing one of the three height-2 sites other than the one at $\vec{i}$, to height 1, will create an FSC outside the five-site cluster, so that Fig. 4 is allowed, but Fig. 5 is not.

However, if we impose the condition that taking the height of $\vec{i}$ from 2 to 1 should produce no FSC larger than the five-site cluster, then all four configurations in Figs. 4 and 5 are equally probable. With this condition, decreasing one of the height-2 sites in Fig. 4 cannot possibly create an FSC outside the five-site cluster, since then the union of this FSC with the five sites would be a larger FSC upon decreasing the height of $\vec{i}$.

For a WAC, it can be shown that if any height is decreased, not only does the WAC become a FSC, but it is not contained in any larger FSC. Therefore, for the three height configurations in Fig. 5, the condition that the five-site FSC generated is maximal is automatic. (However, for the configuration in Fig. 4 it is not.) Therefore the probability associated with the modified sandpile in Fig. 3 is four times the probability of any of the three WACs in Fig. 5. It is also four times the probability of the height configuration in Fig. 4, if we impose on this configuration the condition that decreasing the height of $\vec{i}$ should produce no FSC larger than the five-site cluster (although clearly this is not as physically interesting).

In Ref. [18], the configurations that they labeled S_{10} and S_{11} were not WACs. Once they are modified to be WACs with the same shape, our values for A, B_1, and B_2, obtained with the methods of the previous section, agree with theirs.

Although we chose a specific sandpile modification, the discussion is easily generalized. Generally, consider sandpile modifications similar to those in Fig. 3, which separate a cluster of sites from the rest of the sandpile, except for one linking site. There will be N possible height configurations in

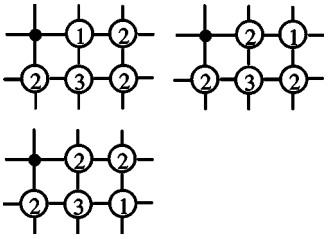

FIG. 5. Three WACs with the same shape as the configuration in Fig. 3.

the cluster that become FSCs if the height of the linking site is reduced. Some of these will be WACs and some will not. The probability associated with the lattice modification will be N times the probability for any individual WAC. (It will also be N times the probability for any of the other height configurations, given the condition that decreasing the height of the link site should produce no FSC larger than the cluster in question.)

These issues did not need to be discussed for the simpler WACs, such as those in Fig. 1 (or the other WACs of Ref. [18]). For each of the sandpile modifications in the right side of Fig. 1, only one corresponding WAC height configuration is possible ($N=1$).

VI. OFF-BOUNDARY EXPECTATION VALUES

These methods allow us to quickly determine the effects of a number of defects or boundary conditions on all LBMs, and see that each time, we obtain the same coefficients. In many cases, a defect or boundary changes the Green function matrix in a manner such that the change factorizes, taking the form $K\vec{f}\vec{g}^T$, for some vectors $\vec{f}$ and $\vec{g}$. Equation (29) then shows that the effects of the change can be written as a linear combination of the c's, and thus as a linear combination of A, B_1, and B_2.

For example, consider the probability for an LBM located a distance y from a boundary (open or closed). Let the boundary be at $y=0$, and x be the coordinate along the boundary. Then the Green function is modified to [22]

$$G_{\text{open}}(x_1,y_1;x_2,y_2) = G_0(x_1-x_2,\ y_1-y_2) - G_0(x_1-x_2,y_1+y_2+2), \quad (30)$$

$$G_{\text{closed}}(x_1,y_1;x_2,y_2) = G_0(x_1-x_2,\ y_1-y_2) + G_0(x_1-x_2,y_1+y_2+1). \quad (31)$$

Placing the local origin of the LBM at $(0,y)$, the Green function between points (k_1,l_1) and (k_2,l_2), relative to this local origin, where k_1, l_1, k_2, and l_2 are all $O(1)$, is

$$G(k_1,l_1;k_2,l_2) = G_0(k_1-k_2,\ l_1-l_2) \pm \frac{k_1k_2+l_1l_2}{8\pi y^2} + O\left(\frac{1}{y^3}\right). \quad (32)$$

In the $\pm$, the top sign is for closed boundaries, and the bottom sign is for open boundaries. As with Eq. (7), we have only kept terms that depend on both (k_1,l_1) and (k_2,l_2) [otherwise, there would be terms of $O(\ln y)$ and $O(1/y)$]. Then, using Eq. (29), keeping only terms of $O(1/y^2)$, and using the definitions in Eqs. (26)–(28) and Eqs. (15)–(17), we immediately see that the probability for the LBM is

$$p_u \mp \frac{A_u}{4y^2} + O\left(\frac{1}{y^3}\right). \quad (33)$$

This agrees with results for the unit height variable found in Ref. [22].

VII. BOND DEFECTS IN THE BULK

We have also used the methods described here to investigate bond defects in the ASM. We change the toppling matrix from the defect-free matrix $\mathbf{\Delta}_0$ to

$$\Delta(\vec{i},\vec{j}) = \Delta_0(\vec{i},\vec{j}) + \delta\Delta(\vec{i},\vec{j}), \quad (34)$$

$$\delta\Delta(\vec{i},\vec{j}) = \begin{cases} -k_{\text{bond}} & \text{if}(\vec{i},\vec{j}) = (\vec{s}_0,\vec{s}_0) \text{ or } (\vec{i},\vec{j}) = (\vec{s}_1,\vec{s}_1) \\ +k_{\text{bond}} & \text{if}(\vec{i},\vec{j}) = (\vec{s}_0,\vec{s}_1) \text{ or } (\vec{i},\vec{j}) = (\vec{s}_1,\vec{s}_0) \end{cases}. \quad (35)$$

If $\vec{s}_0$ and $\vec{s}_1$ are adjacent sites, and $k_{\text{bond}}=1$, then this corresponds to removing the bond between $\vec{s}_0$ and $\vec{s}_1$. If $k_{\text{bond}}=-1$, this corresponds to adding a bond between $\vec{s}_0$ and $\vec{s}_1$.

The Green function is the inverse of the toppling matrix, and the effects of this perturbation can be calculated by summing a geometric series. The result is

$$G(\vec{i},\vec{j}) = G_0(\vec{i},\vec{j}) + \tilde{k}_{\text{bond}}[G_0(\vec{i},\vec{s}_0) - G_0(\vec{i},\vec{s}_1)] \times [G_0(\vec{s}_0,\vec{j}) - G_0(\vec{s}_1,\vec{j})], \quad (36)$$

where

$$\frac{1}{\tilde{k}_{\text{bond}}} - \frac{1}{k_{\text{bond}}} = G_0(\vec{s}_0,\vec{s}_0) + G_0(\vec{s}_1,\vec{s}_1) - G_0(\vec{s}_0,\vec{s}_1) - G_0(\vec{s}_1,\vec{s}_0). \quad (37)$$

The correction to the Green function factorizes, just as in Eq. (29), and the corrections to the LBM probabilities again depend on the same coefficients. If the two ends of the bond defect are at (0, 0) and (q_x,q_y), where q_x and q_y are both $O(1)$, then from Eqs. (15)–(17), Eqs. (26)–(28), and Eq. (29), the probability for a LBM of type u at $(x,y) = (r\cos\theta, r\sin\theta)$, for $r \gg 1$, is

$$-\frac{\tilde{k}_{\text{bond}}}{4\pi r^4}\{(q_x^2+q_y^2)A_u + (q_x^2-q_y^2)[B_{1u}\cos(4\theta) + B_{2u}\sin(4\theta)] + (2q_xq_y)[B_{1u}\sin(4\theta) - B_{2u}\cos(4\theta)]\}. \quad (38)$$

This is consistent with identifying the bond defect with

$$-\frac{\tilde{k}_{\text{bond}}}{2\pi} :(q_x\partial_x\theta + q_y\partial_y\theta)(q_x\partial_x\overline{\theta} + q_y\partial_y\overline{\theta}):. \quad (39)$$

VIII. BOUNDARY OPERATORS AND BOND DEFECTS

It was shown in Ref. [14] that any local arrow diagram along an open boundary is represented in the LCFT by the operator

$$-\frac{2}{\pi}\det(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu})\left((\vec{y}+\vec{1})^T\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}}\mathbf{B}_u(\vec{y}+\vec{1})\right)\times\partial\theta\partial\overline{\theta}, \quad (40)$$

where $\vec{y}$ is the vector of distances perpendicular to the boundary. The arguments there worked for *any* local arrow

diagram (not just those corresponding to LBMs), and similarly to our arguments in Sec. III, used the fact that the Green function along open boundaries falls off as $1/x^2$. Along closed boundaries (and in the bulk), the Green function grows as $\ln x$, so the situation is more complicated, and not all operators are proportional to $\partial\theta\partial\overline{\theta}$. (See, for example, the operators for the height -2 and -3 variables along closed boundaries, given in Refs. [13,14].) However, we can now derive an expression similar to Eq. (40) for LBMs along closed boundaries.

The Green function for two sites on a closed boundary, and $x \gg 1$ apart, was found in Ref. [22], using Eq. (31), to be

$$G_{\text{closed}}(0,0;x,0) = -\frac{1}{\pi}\ln(x) - \left(\frac{\gamma}{\pi} + \frac{\ln 2}{2\pi}\right) + \frac{1}{6\pi x^2} + O\left(\frac{1}{x^4}\right). \tag{41}$$

Using the recursion relation $\mathbf{G\Delta}=\mathbb{I}$ in this equation, we can extend Eq. (41) for points $O(1)$ from the boundary:

$$G_{\text{closed}}(x_1,y_1;x+x_2,\,y_2) = G_{\text{closed}}(0,0;x,0) - \frac{1}{\pi}\ln\left(1+\frac{x_2-x_1}{x}\right) - \frac{y_1(y_1+1)+y_2(y_2+1)}{2\pi(x+x_2-x_1)^2} + \frac{x_2-x_1}{3\pi x^3} + O\left(\frac{1}{x^4}\right). \tag{42}$$

[x_1, x_2, y_1, and y_2 are all $O(1)$.] The Green function diverges as $\ln x$, but if we are calculating correlations of LBMs along closed boundaries, we can use the arguments of Sec. III to see that we only care about the parts of the Green function matrix that depend on both the row and the column indices. The part of Eq. (42) that depends on both (x_1,y_1) and (x_2,y_2) is

$$G_{\text{closed}}(x_1,y_1;x+x_2,y_2) \to -\frac{x_1x_2}{\pi x^2} + \frac{x_1^2x_2 - x_1x_2^2 + x_1y_2(y_2+1) - x_2y_1(y_1+1)}{\pi x^3} + O\left(\frac{1}{x^4}\right). \tag{43}$$

Using logic identical to that in the bulk case, we can use the $O(1/x^2)$ part, to derive field identifications for LBMs along closed boundaries:

$$\frac{2}{\pi}\det(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu})\left(\vec{x}^T\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}}\mathbf{B}_u\vec{x}\right)\partial\theta\partial\overline{\theta}. \tag{44}$$

$\vec{x}$ is the vector of position coordinates parallel to the boundary.

We now introduce a bond defect of strength k_{bond}, along an open or closed boundary, between sites (q_{x1}, q_{y1}) and (q_{x2}, q_{y2}), that are $O(1)$ apart. These bond defects can be analyzed as in the previous section. Along an open boundary, the bond defect is represented by

$$\frac{2}{\pi}\tilde{k}_{\text{bond}}(q_{y1}-q_{y2})^2\partial\theta\partial\overline{\theta}, \tag{45}$$

if $q_{y1} \neq q_{y2}$—i.e., if the bond defect has a vertical component. On the other hand, if the bond defect is purely horizontal, it is represented by

$$\frac{2}{\pi}\tilde{k}_{\text{bond}}(q_{y1}+1)^2(q_{x1}-q_{x2})^2\partial^2\theta\partial^2\overline{\theta}. \tag{46}$$

A purely horizontal bond along an open boundary is represented by a dimension 4 operator.

For closed boundaries, the bond defect is represented by

$$-\frac{2}{\pi}\tilde{k}_{\text{bond}}(q_{x1}-q_{x2})^2\partial\theta\partial\overline{\theta}, \tag{47}$$

if the bond has a horizontal component. If the bond is purely vertical, it is represented by

$$-\frac{\tilde{k}_{\text{bond}}}{2\pi}[q_{y1}(q_{y1}+1)-q_{y2}(q_{y2}+1)]^2\partial^2\theta\partial^2\overline{\theta}. \tag{48}$$

Along closed boundaries, it is the purely vertical bonds that have dimension 4.

We have verified these field identifications of bond defects with more general calculations, involving multiple fields and multiple bond defects. This required generalizing Eq. (36) for multiple bond defects. However, the generalization is straightforward, and not particularly instructive, so is not shown here.

IX. ASM WITH DISSIPATION

We now consider the addition of dissipation. As explained in the introduction, this takes the ASM off the critical point, as shown by both numerical simulations, and an exact analysis [15–17]. The toppling matrix becomes

$$\Delta_{\vec{i},\vec{j}} = \begin{cases} 4+t & \text{if } \vec{i}=\vec{j} \\ -1 & \text{if } \vec{i} \text{ and } \vec{j} \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}. \tag{49}$$

Now, with each toppling in the bulk of the sandpile, t grains of sand are lost.

If we set $t=0$, we get back the original, critical ASM. It should be noted that the interpretation of the continuous $t \to 0$ limit is potentially problematic. The modification to $\mathbf{\Delta}$ described in Eq. (49) is only sensible for integer t. It can be extended to rational t [16]. However, for t rational, but not an integer, the interpretation of the sandpile modifications (i.e., the $\mathbf{B}$ matrix) associated with a WAC is changed, so that what is meant by taking a limit of infinitesimally small, rational t is unclear. However, presumably the fact that the massive results are good for all integers $t>0$ can justify an analytic continuation to $t=0$. Regardless, we are certainly able to formally expand all correlation functions in Taylor series about $t=0$, which is what we do here.

t defines an effective mass M for the sandpile, where $t=a^2M^2$ [18]. a can be thought of as the lattice spacing. In looking at off-critical correlations of LBMs, we are interested in correlation functions where the number of lattice spacings between any two of the LBMs is $O(r/a)$. Taking the $a \to 0$ limit then defines the way in which we simultaneously take $t \to 0$ and distances between LBMs to infinity.

For the off-critical sandpile, as discussed in Ref. [18], we can use the same methods as before to calculate correlations, with two modifications. First, we need to use a different $\mathbf{B}$ matrix than before. Previously, we required that our sandpile modifications be conservative, which meant that each row and each column of $\mathbf{B}$ summed up to zero. However, for the massive sandpile we will often want to consider nonconservative $\mathbf{B}$'s. [For the unit height variable, the sandpile modification in Eq. (3) will no longer restrict the height of $\vec{i}$ to 1, but rather to any height from 1 to $1+t$.] We are most interested in LBMs associated with WACs. If we want to force the heights to the heights of the WAC, the $\mathbf{B}$ matrix must be changed to

$$\mathbf{B} = \mathbf{B}_c - t\mathbf{B}_{nc}. \tag{50}$$

$\mathbf{B}_c$ is the $\mathbf{B}$ matrix that would be used for this WAC for the nondissipative ASM [e.g., Eq. (3)], and

$$(B_{nc})_{\vec{i},\vec{j}} = \begin{cases} 1 \text{ if } \vec{i}=\vec{j}, \text{and } \vec{i} \text{ is in the WAC} \\ \text{height configuration} \\ 0 \text{ otherwise} \end{cases}. \tag{51}$$

By "in the WAC height configuration," we mean in the set of sites whose heights are fixed, and not in one of the bordering sites needed to to form the $\mathbf{B}$ matrix (e.g., for the unit height configuration, only one site is "in the WAC height configuration"). More generally, for other LBMs, $\mathbf{B}_{nc}$ is the nonconservative part of the $\mathbf{B}$ matrix.

Second, we need to use a new Green function between lattice sites. As always, the Green function is given by the inverse of the toppling matrix. In Appendix A, we calculate the Green function in the limit $t \to 0$, when the distance between two sites scales as $1/\sqrt{t}$, and find that it approaches $(1/2\pi)K_0(r)$, where K_0 is the modified Bessel function of the second kind. The asymptotic expansion in $1/\sqrt{t}$ is then

$$G_0\left(\frac{r\cos\phi}{\sqrt{t}}, \frac{r\sin\phi}{\sqrt{t}}\right) \to \frac{1}{2\pi}K_0(r) + \sqrt{t}f_1(r,\phi) + tf_2(r,\phi) + O(t^{3/2}). \tag{52}$$

We have not calculated the f functions, because they turn out to not affect the universal parts of any correlation functions. In principle, they can contain bounded functions of $1/\sqrt{t}$, such as $e^{ir/\sqrt{t}}$, but we have not explicitly indicated this t dependence, since it does not affect our analysis. [Mahieu and Ruelle found, for $\phi=0$ and $\phi=\pi/4$, Eq. (52), and the specific forms of f_1 and f_2 [18]. They found that $f_1=0$ for these angles, so it is possible that $f_1=0$ for all ϕ, although we have not investigated this.]

We decompose $\mathbf{G}_{uv}$ as a sum of four $N_u \times N_v$ matrices:

$$\mathbf{G}_{uv} = \mathbf{G}_{uv,J} + \sqrt{t}\mathbf{G}_{uv,\text{row}} + \sqrt{t}\mathbf{G}_{uv,\text{col}} + t\mathbf{G}_{uv,\text{both}}. \tag{53}$$

$\mathbf{G}_{uv,J}$ is a matrix in which every element is identical. $\mathbf{G}_{uv,\text{row}}$ and $\mathbf{G}_{uv,\text{col}}$ are matrices in which the elements depend only on the row index, or only on the column index. Parts of $\mathbf{G}$ which cannot be written in these forms go into $\mathbf{G}_{uv,\text{both}}$. All four of these matrices are Taylor series in $\sqrt{t}$, whose $O(1)$ terms depend only on the Taylor expansion of $(1/2\pi)K_0(r)$, and whose higher order terms in $\sqrt{t}$ depend on f_1, f_2, etc. For example, every element of $\mathbf{G}_{uv,J}$ is $(1/2\pi)K_0(r_{uv}) + \sqrt{t}f_1(r_{uv},\phi_{uv}) + \ldots$. The elements of $\mathbf{G}_{uv,\text{row}}$ and $\mathbf{G}_{uv,\text{col}}$ depend on only the coordinates in the uth LBM, or on only the coordinates in the vth LBM, and thus require one derivative (finite difference) of the Green function. The elements of $\mathbf{G}_{uv,\text{both}}$ require two or more derivatives of the Green function.

For correlations at the critical point, we saw that $\mathbf{G}_{uv,\text{J}}$, $\mathbf{G}_{uv,\text{row}}$, and $\mathbf{G}_{uv,\text{col}}$ could all be ignored in calculating LBM correlations. However, the arguments there relied on the fact that every row and every column of every $\mathbf{B}_u$ summed to zero. That no longer holds here, and we thus need to reconsider which terms of the Green function we need to keep.

We expand the correlation functions in powers of t, and look for the lowest, nonzero, power of t. As with critical correlations, to prove the validity of Eq. (10), we need to show that the lowest-order term of the n-point function only comes from the parts of $\det(\mathbb{I}+\mathbf{BG})$ with n terms off the block diagonal. However, the proof is much harder in this case; we sketch the proof in Appendix B.

Similarly to Eq. (7), we need to Taylor expand

$$G_0\left(\frac{1}{\sqrt{t}}r_{uv}\cos\phi_{uv} + k_2 - k_1, \frac{1}{\sqrt{t}}r_{uv}\sin\phi_{uv} + l_2 - l_1\right). \tag{54}$$

Then, just as in Eq. (12), we can write $\mathbf{G}_{uv}$ as a sum of terms, each of which is the product of a length N_u column vector and a length N_v row vector. Defining $\vec{1}_u$ to be the vector of length N_u, all of whose entries are 1, we have

$$2\pi\mathbf{G}_{uv} = K_0(r_{uv})\vec{1}_u\vec{1}_v^T + [K_0'(r_{uv})\cos\phi_{uv}\sqrt{t}](\vec{1}_u\vec{k}_v^T - \vec{k}_u\vec{1}_v^T) + [K_0'(r_{uv})\sin\phi_{uv}\sqrt{t}](\vec{1}_u\vec{l}_v^T - \vec{l}_u\vec{1}_v^T) - [K_0(r_{uv})t] \times\Big[\sin^2\phi_{uv}\vec{k}_u\vec{k}_v^T + \cos^2\phi_{uv}\vec{l}_u\vec{l}_v^T + \tfrac{1}{2}\sin(2\phi_{uv})(\vec{k}_u\vec{l}_v^T + \vec{l}_u\vec{k}_v^T)\Big] + (K_0''(r_{uv})t)[\cos(2\phi_{uv})(\vec{l}_u\vec{l}_v^T - \vec{k}_u\vec{k}_v^T) - \sin(2\phi_{uv})(\vec{k}_u\vec{l}_v^T + \vec{k}_v\vec{l}_u^T)] + \cdots . \quad (55)$$

In the ellipses, we have dropped not only terms of $O(t^{3/2})$ and higher, but all terms with f_1 or f_2. The terms with f_1 or f_2 are not necessarily higher order in t than the terms shown. f_1 contributes terms of $O(\sqrt{t})$ to $\vec{1}_u\vec{1}_v^T$, and terms of $O(t)$ to $\vec{1}_u\vec{k}_v^T, \vec{1}_u\vec{l}_v^T, \vec{k}_u\vec{1}_v^T$, and $\vec{l}_u\vec{1}_v^T$. f_2 contributes terms of $O(t)$ to $\vec{1}_u\vec{1}_v^T$. Similarly to Eq. (14), we represent $\mathbf{G}_{uv}$ with a 3×3 matrix, $\mathbf{N}'_{uv}$, where each element of $\mathbf{N}'_{uv}$ represents a different choice of row vector and column vector:

$$\mathbf{N}'_{uv} \equiv \frac{1}{2\pi}\begin{array}{c} \begin{array}{ccc} \vec{k}_v^T & \vec{l}_v^T & \vec{1}_v^T \end{array} \\ \begin{pmatrix} -t[\sin^2\phi K_0 + \cos(2\phi)K_0''] + O(t^{3/2}) & (t/2)[\sin(2\phi)(K_0 - 2K_0'')] + O(t^{3/2}) & -K_0'\cos\phi\sqrt{t} + O(t) \\ (t/2)\sin(2\phi)(K_0 - 2K_0'') + O(t^{3/2}) & -t[\cos^2\phi K_0 - \cos(2\phi)K_0''] + O(t^{3/2}) & -K_0'\sin\phi\sqrt{t} + O(t) \\ K_0'\cos\phi\sqrt{t} + O(t) & K_0'\sin\phi\sqrt{t} + O(t) & K_0 + O(t^{1/2}) \end{pmatrix} \end{array} \begin{array}{c} \vec{k}_u \\ \vec{l}_u \\ \vec{1}_u \end{array} . \quad (56)$$

To save space, we have here abbreviated $\phi_{uv}\to\phi$ and $K_0(r_{uv})\to K_0$. Now, while some of the terms depending on f_1 and f_2 are $O(\sqrt{t})$ and $O(t)$, they are higher order terms in $\mathbf{G}_{uv,\mathrm{J}}$, $\mathbf{G}_{uv,\mathrm{row}}$, $\mathbf{G}_{uv,\mathrm{col}}$, and $\mathbf{G}_{uv,\mathrm{both}}$, and thus are higher order in the specific matrix elements of $\mathbf{N}'_{uv}$ that they contribute to. We will later see that this justifies dropping them.

Just as in Sec. III, when each off-diagonal Green function matrix is replaced by the product of a column vector and a row vector, each $(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu})^{-1}\mathbf{B}_u$ is bracketed by a row vector to its left and a column vector to its right, producing a 1×1 matrix. We thus get a matrix, $\mathbf{M}'_u$, similar to the $\mathbf{M}_u$ of Eq. (13). $\mathbf{M}'_u$ is 3×3, rather than 2×2, because the vectors bracketing $(\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu})^{-1}\mathbf{B}_u$ to the left and to the right can now be $\vec{k}_u, \vec{l}_u$, or $\vec{1}_u$.

In principle, when calculating $\mathbf{M}'_u, \mathbf{B}_u$, and $\mathbf{G}_{uu}$ should be the matrices for the massive sandpile. However, we only need most elements of $\mathbf{M}'_u$ in the limit $t\to0$ (this will be justified shortly). In this limit, we replace $\mathbf{B}_u$ with $\mathbf{B}_{u,c}$, and the elements of $\mathbf{G}_{uu}$ with the normal, well-known, critical Green function. Thus, to lowest-order, the elements of $\mathbf{M}_u$ in $\mathbf{M}'_u$ are unchanged:

$$c'_{u,kk} = c_{u,kk} + O(t), \quad (57)$$

$$c'_{u,kl} = c_{u,kl} + O(t), \quad (58)$$

$$c'_{u,ll} = c_{u,ll} + O(t). \quad (59)$$

For the new entries of $\mathbf{M}'_u$, it is not hard to show that

$$c'_{u,1k} \equiv -p_u\vec{1}_u^T\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}}\mathbf{B}_u\vec{k}_u = O(t), \quad (60)$$

$$c'_{u,1l} \equiv -p_u\vec{1}_u^T\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}}\mathbf{B}_u\vec{l}_u = O(t), \quad (61)$$

$$c'_{u,11} \equiv -p_u\vec{1}_u^T\frac{\mathbb{I}}{\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}}\mathbf{B}_u\vec{1}_u = -p_u\vec{1}_u^T(-t\mathbf{B}_{u,nc})\vec{1}_u + O(t^2) = tp_uC + O(t^2), \quad (62)$$

where p_u is the probability for the LBM, and C is defined as the number of sites in the WAC height configuration [as defined below Eq. (51)]. $c'_{u,11}$ is the only element of $\mathbf{M}'_u$ where we need the $O(t)$ term. Mahieu and Ruelle defined C as the coefficient in Eq. (1), and observed that for the 14 WACs that they considered, C always turned out to always be equal to the number of sites in the WAC height configuration [18]. We will see that our C is the same as their C, proving that their observation holds for all WACs.

We now have

$$\mathbf{M}'_u \equiv \begin{pmatrix} c_{u,kk} + O(t) & c_{u,kl} + O(t) & O(t) \\ c_{u,kl} + O(t) & c_{u,ll} + O(t) & O(t) \\ O(t) & O(t) & tp_uC + O(t^2) \end{pmatrix}. \quad (63)$$

The n-point correlation is given by

$$-\mathrm{Tr}(\mathbf{M}'_{u_1}\mathbf{N}'_{u_1u_2}\mathbf{M}'_{u_2}\mathbf{N}'_{u_2u_3}\cdots\mathbf{M}'_{u_n}\mathbf{N}'_{u_nu_1}) - \{[(n-1)!-1]\ (\text{other trace terms})\}. \quad (64)$$

It is easy to now verify that the higher-order terms in t in $\mathbf{M}'_u$ and $\mathbf{N}'_{uv}$ that we dropped in Eqs. (56) and (63) indeed give contributions of $O(t^{n+1/2})$ or higher to the n-point correlation, justifying the approximations used. We now have all correlations of LBMs in the ASM with dissipation.

We can show that the correlations we have found are the same as the correlations of the field in Eq. (1). Mahieu and Ruelle proposed that the appropriate massive extension of the LCFT in Eq. (20) is [18]

$$S = \frac{1}{\pi}\int d^2x \left\{ :\partial\theta\bar{\partial}\bar{\theta}: + \frac{M^2}{4}:\theta\bar{\theta}: \right\}. \tag{65}$$

This theory is still Gaussian, with correlation functions

$$\langle \theta(z_u)\bar{\theta}(z_v)\rangle = K_0(M|z_u - z_v|), \tag{66}$$

$$\langle \theta(z_u)\theta(z_v)\rangle = 0, \tag{67}$$

$$\langle \bar{\theta}(z_u)\bar{\theta}(z_v)\rangle = 0. \tag{68}$$

Other two-point correlations can then be obtained by taking holomorphic or antiholomorphic derivatives. Then, just as in Sec. III, since the theory is Gaussian, we can write the n-point correlation of Eq. (1) exactly. The result is formally identical to Eq. (25), where the analogues of **F** and **H** in Eqs. (23) and (24) are now 3×3 [since the θ and $\bar{\theta}$ fields in Eq. (1) may have holomorphic derivatives, antiholomorphic derivatives, or no derivatives at all]. Using the identifications in Eqs. (26)–(28), and $t=a^2M^2$, the results agree with the correlation function in Eq. (64) (up to a proportionality factor, a^{2n}).

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APPENDIX A: MASSIVE GREEN FUNCTION

The Green function for the ASM with dissipation is the inverse of the toppling matrix $\mathbf{\Delta}$, given in Eq. (49). In the limit of an infinite lattice, this can be found by Fourier transform [18]:

$$G_0(m,n) = \int_{-\pi}^{\pi}\frac{dp_x}{2\pi}\int_{-\pi}^{\pi}\frac{dp_y}{2\pi}\frac{e^{i(p_x m + p_y n)}}{4 + t - 2\cos p_x - 2\cos p_y}. \tag{A1}$$

We are interested in this integral either for m and n both $O(1)$, or for m and n both proportional to $1/\sqrt{t}$, in the limit $t\to 0$.

We saw in Sec. IX, that for m and n both $O(1)$, we only needed $G(m,n)-G(0,0)$, in the limit $t\to 0$. But this is just the standard, dissipation-free, lattice Green function, whose properties can be looked up in standard references—for example, Ref. [21]. So we only need to discuss the case where m and n are both proportional to $1/\sqrt{t}$:

$$\lim_{t\to 0} G_0\left(\frac{r\cos\phi}{\sqrt{t}}, \frac{r\sin\phi}{\sqrt{t}}\right) = \lim_{t\to 0}\int_{-\pi}^{\pi}\frac{dp_x}{2\pi}\int_{-\pi}^{\pi}\frac{dp_y}{2\pi} \times \frac{e^{i(p_x\cos\phi + p_y\sin\phi)r/\sqrt{t}}}{4 + t - 2\cos p_x - 2\cos p_y}. \tag{A2}$$

In the limit $t\to 0$, the exponential oscillates infinitely rapidly, so when multiplied by any function smooth in the limit $t\to 0$, gives an integral of zero. So no error is introduced by changing the region of integration to a small disc of radius ϵ about $(p_x,p_y)=(0,0)$. Similarly, because

$$\frac{1}{4 + t - 2\cos p_x - 2\cos p_y} - \frac{1}{p_x^2 + p_y^2 + t} \tag{A3}$$

is smooth over this region, we can replace the first term with the second one. Changing to polar coordinates, the integral is

$$\lim_{t\to 0}\int_0^{\epsilon} p\,dp\int_0^{2\pi} d\alpha \frac{e^{i(\cos\phi\cos\alpha + \sin\phi\sin\alpha)rp/\sqrt{t}}}{(2\pi)^2(p^2 + t)}. \tag{A4}$$

With a change of variables, the integral becomes

$$\lim_{t\to 0} G_0\left(\frac{r\cos\phi}{\sqrt{t}}, \frac{r\sin\phi}{\sqrt{t}}\right) = \frac{1}{(2\pi)^2}\int_0^{\infty}\frac{p\,dp}{p^2 + 1}\int_0^{2\pi} d\alpha e^{irp\cos\alpha} = \frac{1}{2\pi}K_0(r), \tag{A5}$$

where K_0 is the modified Bessel function of the second kind.

To find the leading-order critical limit, we take $r\to 0$ (while $r/\sqrt{t}$ is still large), and use $K_o(r)\to -\ln(r)$, reproducing the first term of Eq. (6).

APPENDIX B: THE TRACE FORMULA FOR MASSIVE CORRELATIONS

In this appendix, we sketch the proof that the trace formula in Eq. (10) is valid for all n-point correlations of LBMs off the critical point. This requires showing that in the determinant $\det(\mathbb{I}+\mathbf{BG})$, if we only want terms up to $O(t^n)$, we never need to pick more than n terms off the block diagonal of $\mathbb{I}+\mathbf{BG}$. In other words, when we calculate the determinant with

$$\det \mathbf{X} = \sum_{p\in S_{|X|}} (-)^p X_{1,p(1)}X_{2,p(2)}\cdots X_{n,p(n)}, \tag{B1}$$

where p is summed over all permutations of $\{1,2,\ldots,|X|\}$, we never have more than n of the $X_{i,p(i)}$'s from the off-diagonal blocks. The off-diagonal blocks of $\mathbb{I}+\mathbf{BG}$ all have the form $\mathbf{B}_u\mathbf{G}_{uv}, u\neq v$, where $\mathbf{B}_u$ and $\mathbf{G}_{uv}$ can be written with Eqs. (50) and (53). Since every row of $\mathbf{B}_{c,u}$ sums to zero, $\mathbf{B}_{c,u}\mathbf{G}_{uv,J}=\mathbf{0}$, and $\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{col}}=\mathbf{0}$. The off-diagonal block can thus be written

$$\mathbf{B}_u\mathbf{G}_{uv} = \sqrt{t}\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{row}} + t\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{both}} - t\mathbf{B}_{nc,u}\mathbf{G}_{uv,J} + O(t^{3/2}). \tag{B2}$$

[We need the $O(t^{3/2})$ terms for the calculation of the correlation functions in Sec. IX, but do not need their explicit form for this proof.] Since there are terms of $O(\sqrt{t})$ in the off-diagonal blocks, naively, to get the $O(t^n)$ contribution to the correlation function, we would need parts of the determinant with up to $2n$ terms off the block diagonal. So we need to explain why the terms with more than n terms off the block diagonal in fact have all their contributions to the $O(t^n)$ part of the correlation function cancel (as well as why all the terms with lower powers of t cancel).

We define a "row matrix" to be a matrix in which the entries depend only on the row index. When we consider

contributions to det($\mathbb{I}+\mathbf{BG}$), we consider, for each contributing matrix element off the block diagonal, whether it is from $\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{row}}$, $\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{both}}$, $\mathbf{B}_{nc,u}\mathbf{G}_{uv,J}$, or from the elements of $O(t^{3/2})$ or higher. We use several matrix theorems; the first two are general, not referring specifically to **B** or **G**. It is not hard to show:

Theorem 1. The determinant has zero contribution from terms that have both a matrix element from a row matrix in the (u,v_1) block, and a matrix element from a row matrix in the (u,v_2) block.

By "zero contribution," we mean that while specific terms in Eq. (B1) may be nonzero, when all such terms are considered, they cancel. Theorem 1 is easy to prove: if $X_{\alpha\beta}$ is the element from the (u,v_1) block, and $X_{\gamma\delta}$ is the element from the (u,v_2) block, we get a cancelling contribution from $X_{\alpha\delta}$ and $X_{\gamma\beta}$. It is somewhat harder to show the following theorem, which we state without proof:

Theorem 2. Suppose the (u,v) block $(u\neq v)$ is a row matrix, every column of the (v,v) block sums to one, and for every $u'\neq v$, every column of the (v,u') block sums to zero. Then the determinant has zero contribution from the matrix elements of the (u,v) block; in other words, the determinant is unchanged if every element of the (u,v) block is set to zero.

We note that $\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{row}}$ and $\mathbf{B}_{nc,u}\mathbf{G}_{uv,J}$ are both row matrices, and that every column of $\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{row}}$ sums to zero, as does every column of $\mathbf{B}_{c,u}\mathbf{G}_{uv,\mathrm{both}}$. Then, repeatedly applying theorems 1 and 2 allows us to prove the following:

Theorem 3. In the determinant det($\mathbb{I}+\mathbf{BG}$), suppose the (u_0,u_1) block $(u_0\neq u_1)$ has a contributing matrix element from $\mathbf{B}_{c,u_0}\mathbf{G}_{u_0u_1,\mathrm{row}}$. Then, in the terms that produce a nonzero contribution to the determinant, there is an ordered sequence of distinct block indices, $(u_1,u_2,\ldots,u_x)$, $x\geq 1$, such that for all $1\leq i<x$, the (u_i,u_{i+1}) block has a term from $-t\mathbf{B}_{nc,u_i}\mathbf{G}_{u_iu_{i+1},J}$. Furthermore, either

(1) the (u_x,u_x) block has a term of $O(t)$ or higher, or

(2) there is a term of order $O(t^{3/2})$ or higher in an off-diagonal block, (u_x,u_{x+1}).

A different (v_0,v_1) block $(v_0\neq v_1)$ with a matrix element from $\mathbf{B}_{c,v_0}\mathbf{G}_{v_0v_1,\mathrm{row}}$ will produce an ordered sequence of distinct block indices, $(v_1,v_2,\ldots v_y)$, with no elements in common with $(u_1,u_2,\ldots u_x)$.

Next, for any nonvanishing contribution to the determinant, we define

$$c_1 = \text{number of terms off the block diagonal}, \tag{B3}$$

$$c_2 = \text{number of terms off the block diagonal that are exactly } O(t^{1/2}), \tag{B4}$$

$$c_3 = \text{number of terms off the block diagonal that are } O(t^{3/2}) \text{ or higher}, \tag{B5}$$

$$c_4 = \text{number of terms on the block diagonal that are } O(t) \text{ or higher}. \tag{B6}$$

Theorem 3 shows that each term in the determinant of type c_2 can be associated with a distinct term of type c_3 or c_4, so that $c_3+c_4\geq c_2$. The number of powers of t from this contribution to the determinant is then *at least*

$$(c_1-c_2-c_3)(1)+c_2(1/2)+c_3(3/2)+c_4(1)\geq c_1. \tag{B7}$$

So if we only want $O(t^n)$ contributions to the correlation function we should never have more than n terms off the block diagonal. Furthermore, we want a connected correlation function, so we should always have exactly n terms off the block diagonal. This concludes the proof that Eq. (10) is valid for off-critical n-point correlations.

To get strict equality in Eq. (B7), we need $c_1=n$, $c_4=0$, and $c_2=c_3$. Furthermore, the terms of type c_3 should be exactly proportional to $t^{3/2}$. The fact that $c_4=0$ means that in the diagonal blocks, $\mathbb{I}+\mathbf{B}_u\mathbf{G}_{uu}$, we can set $t=0$ at the start of our calculations, as already seen by other means in Sec. IX.

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